A Lagrangian Stochastic Model of Surfzone Drifter Dispersion

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Abstract.

Drifter-derived cross- and alongshore surfzone diffusivities were previously estimated on an alongshore uniform beach over 1000 s for 5 Huntington Beach, CA 2006 (HB06) experiment release days. The cross-shore diffusivity $K_x$ had a non-monotonic time-dependence, potentially due to the shoreline or to weaker diffusivity seaward of the surfzone. The alongshore diffusivities $K_y$ were qualitatively consistent with shear dispersion but differed from the classic Taylor laminar theory. Here, modeled and analytic diffusivities for the 5 release days are derived from a Lagrangian stochastic model (LSM) that uses the drifter-derived bulk (cross-shore averaged) velocity variance and cross-shore dependent mean alongshore current. The LSM modeled and analytic cross-shore diffusivities are non-monotonic due to the shoreline and strongly suggest that the observed cross-shore diffusivity is shoreline affected. The LSM typically reproduce well the observed $K_x$ with Lagrangian time-scale between 75–200 s, consistent with surfzone eddy time-scales. HB06 drifter trajectories were too short to observe the analytic long-time $K_x$ limit, and weaker diffusivity seaward of the surfzone may be important at longer times ($> 1000$ s).

On all release days, the LSM model and analytic alongshore diffusivity reproduces well the observed $K_y$ with alongshore Lagrangian time-scales between 95–155 s. The isolated shear-induced diffusivity is very well represented by an analytic theory which incorporates a non-zero Lagrangian time-scale. Many of the the stochastic model parameters can be specified apriori with
reasonable assumptions to predict surfzone dispersion of an initial value problem pollution spill.
1. Introduction

Elevated levels of surfzone contaminants, whether fecal indicator bacteria [Reeves et al., 2004] or viruses [Jiang and Chu, 2004], increase health risks to ocean bathers [Haile et al., 1999] and lead to beach closures [Boehm et al., 2002]. The surfzone is also habitat to ecologically important species of fish, invertebrates, and macro-algae [e.g., Brown and McLachlan, 2002]. Understanding tracer transport and diffusion (dilution) is critical for predicting beach water quality or for larval recruitment. For a known surfzone source, tracer (pollution, larvae, etc.) transport up or down coast is understood for simple relatively alongshore uniform beaches [Ruessink et al., 2001; Grant et al., 2005]. However, surfzone diffusion, which impacts the fate of surfzone fecal indicator bacteria [Rippy et al., 2011], is not as well understood.

Surfzone drifters have been used both experimentally [Spydell et al., 2007, 2009; Brown et al., 2009] and numerically [Spydell and Feddersen, 2009; Geiman et al., 2011] to estimate cross- and alongshore surfzone absolute diffusivities and to investigate the mechanisms driving drifter dispersion. Similarly, surfzone dye observations [Clark et al., 2010] and numerical simulations [Clark et al., 2011] have also been used to estimate surfzone tracer dispersion and its mechanisms. For times greater than a few wave periods, drifter and dye cross-shore dispersion is due to low frequency (< 0.03 Hz) surfzone eddies [Spydell and Feddersen, 2009; Clark et al., 2010, 2011].

Lagrangian-derived diffusivities are time-dependent. In homogeneous turbulence, the diffusivity $K$ is monotonic with a linear ballistic regime ($K = \overline{u^2}t$, where $t$ is time and $\overline{u^2}$ is the velocity variance) for times much less than the Lagrangian time-scale $\tau$ and become
constant \( K = \bar{u}^2 \tau \) at times much larger than \( \tau \) \cite{Taylor, 1922}. For five drifter release-days on an alongshore uniform beach, the cross-shore \((x)\) and alongshore \((y)\) diffusivities \((K_x, K_y)\), respectively) were estimated for times less than 1000 s using an unbiased estimator \cite{Spydell et al., 2009}.

The observed \( K_x \) had a ballistic regime for \( t < 50 \text{ s} \), a \( K_x \) maximum \( \approx 2 \text{ m}^2\text{s}^{-1} \) around 150–300 s, with slow decay for longer times (Figure 1a). The time to reach maximum and the long-time decay rate varied with release day. This non-monotonic \( K_x \) pattern was not generally observed by \textit{Spydell et al.} \cite{Spydell et al., 2007} on another alongshore uniform beach because biased estimators were used. On a rip-channeled beach, non-monotonic \( K_x \) time-dependence also was observed \cite{Brown et al., 2009}, although this was linked to the transit time through the rip-circulation cells.

A non-monotonic surfzone \( K_x \) may occur either because the shoreline boundary prevents unbounded diffusion or because diffusion is weaker seaward of the surfzone. Consistent with weaker seaward of the surfzone diffusivity, observed and modeled low-frequency rotational velocities (eddy energy) are significantly weaker just 60 m seaward of the surfzone \cite{Feddersen et al., 2011}. At short times, both \textit{Spydell et al.} \cite{Spydell et al., 2007} and \textit{Brown et al.} \cite{Brown et al., 2009} found \( K_x \) weaker seaward of the surfzone relative to within the surfzone. However at longer times, larger sampling errors and biases prevent determining if \( K_x \) was smaller seaward of the surfzone. \textit{Brown et al.} \cite{Brown et al., 2009} observed that very few drifters left the surfzone during their deployments, suggesting weak diffusion offshore. Similarly, \textit{Spydell et al.} \cite{Spydell et al., 2009} also found that surfzone-released drifters rarely went more than 60 m offshore of the surfzone during 10–20 min releases. \textit{Reniers et al.} \cite{Reniers et al., 2009} found that including low-
frequency eddies was crucial to reproducing the observed [Brown et al., 2009] surfzone drifter ejection statistics.

The observed alongshore diffusivity $K_y$, for the Huntington Beach, CA 2006 experiment (HB06) drifters [Spydell et al., 2009], is larger than $K_x$ and grows monotonically (Figure 1b). The ramp-up time-scales are large (> 340 s) for 3 of release days, and for the 1000 s of observations, the observed $K_y$ did not always asymptote [Spydell et al., 2009]. The inferred long-time alongshore diffusivity $K_y$ was proportional to the maximum alongshore current squared, qualitatively consistent with the Taylor [1953] theory of laminar shear dispersion in a pipe, but differing by a factor of 3. However, the Taylor [1953] theory, which effectively assumes a zero Lagrangian time-scale, may not be applicable in the surfzone where the Lagrangian time-scale (i.e., time-correlated Lagrangian velocities) is non-zero owing to finite-sized horizontal eddies [Spydell and Feddersen, 2011]. The relative contribution of eddy-induced (random) dispersion and shear-dispersion on the surfzone alongshore diffusivity is not known.

Modeled surfzone drifter diffusivities (and other statistics) can be derived from numerical drifters seeded into wave-resolving Boussinesq models [Spydell and Feddersen, 2009; Geiman et al., 2011] or wave-averaged circulation models [Reniers et al., 2009]. In these models, a rich wave-number spectrum of rotational (eddies) velocities advect numerical drifters [e.g., Spydell and Feddersen, 2009], which can complicate diagnosing the causes of the observed time-dependent $K_x$ and $K_y$ behavior.

A simpler approach is to simulate Lagrangian particle trajectories (similar to a random walk) with a Lagrangian stochastic model (LSM) using Langevin equations to represent particle position and velocity. LSMs have been used extensively to study dispersion in ho-
mogeneous turbulence [e.g., Rodean, 1996], two-dimensional turbulence [Pasquero et al., 2001], the atmospheric boundary layer [Wilson and Sawford, 1996], basin scale oceanic dispersion [Berloff and McWilliams, 2002], coastal oceanography [Brickman and Smith, 2002], and the dispersion of coastal larvae [Siegel et al., 2003]. For some applications the LSM can be solved analytically [Wilson et al., 2003] leading to precise time-dependent diffusivities.

Here, using mean current, bulk velocity variance, and Lagrangian time-scale as input, a LSM that includes a shoreline boundary is used to investigate the features of the HB06 observed cross- ($K_x$) and alongshore ($K_y$) diffusivities (Figure 1). The HB06 drifter observations and the diffusivity estimation methods are described in Section 2. Langevin equations are presented Section 3.1 and some analytic solutions to them are given in Section 3.2. Section 3.3 explains how the HB06 drifter data and statistics are modeled using the LSM. The modeled and analytic cross-shore diffusivity $K_x$ are shown to reproduce the time-dependent observed $K_x(t)$ (Section 4.1) and suggests that the non-monotonic observed diffusivity is due to shoreline effects. Also, the modeled and observed along-shore diffusivities $K_y(t)$ agree very well (Section 4.2). The isolated modeled and observed shear-induced diffusivity agree very well with the analytic shear-dispersion solution that includes a non-zero Lagrangian time-scale [Spydell and Feddersen, 2011]. On the release days with stronger alongshore currents (and shear), shear-dispersion is the dominant contribution to $K_y(t)$. The results, assumptions, and implications are discussed in Section 5. In particular, without apriori knowledge of the model input parameters, guidance is provided for parameterizing these quantities for predicting the evolution of the initial value.
problem of a surfzone contaminant spill (Section 5.3). The results are summarized in
Section 6.

2. HB06 Drifter Observations

Surfzone drifter release experiments were performed near Huntington Beach CA as part
of the Fall 2006 HB06 experiment [Spydell et al., 2009]. Relevant details of the experiments
are briefly described here. The cross-shore coordinate $x$ increases negatively offshore
($x = 0$ m is at the mean shoreline) and the alongshore coordinate $y$ increases upcoast.
The bathymetry was approximately alongshore uniform. Ten, 50-cm tall, surfzone GPS-
tracked drifters, that track cross- and alongshore positions $(x(t), y(t))$ at 1 Hz [Schmidt
et al., 2003], were deployed on five release days (Sep. 17, Oct. 2, 3, 14, and Oct. 15,
2006) with variable incident wave and mean current conditions (Table 1). Using values
in Spydell et al. [2009], the surfzone width $L_{sz}$ was generally between 75 m to 100 m
from the shoreline (Table 1). Drifters were released repeatedly within or near the
surfzone, and allowed to drift freely for 15-30 minutes before being collected and re-
released. Drifter tracks suggest advection by alongshore currents and the presence of low-
frequency ($f < 0.03$ Hz) eddies. Drifters rarely advected more than 160 m offshore from
the shoreline. Due to their finite depth, some GPS-tracked drifters were beached at the
shoreline and needed to be re-released farther offshore. In such cases, the drifter track was
terminated at the shoreline ($i.e.,$ absorbing the drifter) and the drifter was re-released
in deeper water. On the other hand, some drifters were not completely beached and
“reflected” off of the shoreline. Thus, the shoreline boundary condition for the observed
drifters is mixed, partially absorbing and partially reflecting.
As the irrotational (sea-swell band) surface gravity wave motions are not directly responsible for surfzone dispersion [Spydell and Feddersen, 2009], the drifter positions are wave-averaged using a Gaussian filter with a low-pass cutoff of 0.033 Hz. From these low-passed drifter positions, drifter velocity time-series \((\dot{x}(t), \dot{y}(t))\) are calculated as finite differences, i.e.,

\[
\dot{x}(t) = \frac{x(t + \Delta t) - x(t)}{\Delta t},
\]

where \(t\) is time and \(\Delta t\) is the sampling interval.

For each release, the cross-shore drifter position probability distribution function (pdf) \(p(x)\), the drifter-derived quasi-Eulerian mean alongshore current \(V(x)\), and velocity variances \((u^2(x)\) and \(v^2(x)\)) are estimated in 16–25 m wide (depending on release day) cross-shore bins. The averaging is over all times and drifters within a bin. The velocity variances are considered quasi-Eulerian as binned Lagrangian statistics can differ from Eulerian ones [e.g., Davis, 1991]. However, the cross-shore structure of these drifter-derived binned velocity statistics are similar to velocity statistics observed from a cross-shore array of Acoustic Doppler Velocimeters (ADV) [Spydell et al., 2009].

The drifter cross-shore position pdf \(p(x)\) was non-uniform across the surfzone with a maximum generally within the surfzone (Figure 2a), indicating that drifters were more likely to sample the mid-surfzone than seaward of the surfzone \((x < -100 \text{ m})\) or near the shoreline \((x \approx 0 \text{ m})\). Three drifter release days (09/17, 10/14, and 10/15) had strong positive mean alongshore currents \(V(x)\) with maximum velocity between 0.3–0.6 ms\(^{-1}\) and were significantly sheared (Figure 2b). In contrast, two release days (10/02 and 10/03) had negative and weak \(V(x)\) \((\approx -0.2 \text{ ms}^{-1})\) with limited velocity shear (Figure 2b). On each day the cross-shore velocity variance \(u^2\) has a maximum around 0.02–0.04 m\(^2\)s\(^{-2}\).
in the mid-surfzone \((x \approx -50 \text{ m in Figure 2c})\). The alongshore velocity variance \(v^2\) has similar magnitude to the cross-shore ones (Figure 2d), with typically less cross-shore structure than \(u^2\) and a maximum generally at or closer to the shoreline. The cross-shore structure of the low-frequency Lagrangian velocity variances are consistent with the Eulerian low-frequency velocity variances observed on different days [Feddersen et al., 2011].

Following Spydell et al. [2009], drifter-derived diffusivities are estimated using the unbiased Lagrangian velocity auto-covariance, defined for cross-shore velocities as \(C_x(t)\),

\[
C_x(t) = \langle \dot{x}(a + t) \dot{x}(a) \rangle - \langle \dot{x}(a + t) \rangle \langle \dot{x}(a) \rangle .
\] (2)

where \(\langle \rangle\) represents an average over all available time-lags \(t\) on a trajectory and an average over all trajectories. Averaging over all possible time lags \(t\) on a trajectory, \(i.e.,\) averaging over \(a\), assumes that a trajectory’s velocity is stationary. On the HB06 drifter release days, the cross- \((K_x(t))\) and alongshore \((K_y(t))\) diffusivities (Figure 1) are estimated from the time-integral of the Lagrangian velocity auto-covariance function \([i.e., \ Taylor, 1922]\)

\[
K_x(t) = \int_0^t C_x(t') \, dt'.
\] (3)

Although surfzone statistics are inhomogeneous and depend on cross-shore position, the diffusivity is calculated as if the statistics were homogeneous, hence \(K_x\) and \(K_y\) are “bulk” quantities which average over all the various cross-shore dependent motions.

### 3. The Lagrangian Stochastic Model

In this section, the Lagrangian stochastic model (LSM) is introduced. In Section 3.1, the Langevin equations for particle motion are stated. In order to gain some perspective
into the properties and statistics of the LSM, analytic solutions to the Langevin equations are presented in Section 3.2. Section 3.3 explains how the observed HB06 drifter data is simulated using the LSM.

### 3.1. The Langevin Equations

Particle trajectories are modeled using the Langevin equations for particle velocity and position,

\[
\frac{du}{dt} = \frac{u}{\tau_x} + \sqrt{\frac{2\sigma_u^2}{\tau_x} w_x} \\
\frac{dv}{dt} = \frac{v}{\tau_y} + \sqrt{\frac{2\sigma_v^2}{\tau_y} w_y} \\
\frac{dx}{dt} = u \\
\frac{dy}{dt} = V(x) + v,
\]

where \((x, y)\) refers to the particle position. Two-dimensional particle motion is considered because the drifter observations are two-dimensional. In addition, the surfzone is a shallow water fluid with depth \(< 3\) m much less than the horizontal scales \(> 10\) m) of dispersion, and where strong breaking-wave generated turbulence \([e.g., Feddersen, 2011]\) leads to strong vertical mixing. Furthermore, two-dimensional models of tracer dispersion can explain observed surfzone cross-shore dye tracer dispersion \([Clark et al., 2010, 2011]\).

The particle’s random cross- and alongshore velocities are \(u\) and \(v\), respectively, with variances denoted by \(\sigma_u^2\) and \(\sigma_v^2\). Particle velocities are correlated in time with the memory of prior velocity over the specified Lagrangian time-scale \(\tau_{x,y}\). Although inhomogeneous Lagrangian velocity variance and time-scales can be included \([e.g., Wilson and Sawford, 1996]\), for simplicity, cross-shore homogeneous (spatially uniform) statistics are considered.
here. The cross-shore sheared mean alongshore current $V(x)$ in (4d) advects particles.

The cross-shore mean Lagrangian current (the sum of onshore Stokes drift and offshore Eulerian return flow) must be zero to prevent accumulation of mass at the shoreline. Thus the model (4) can be thought of as representing depth integrated Lagrangian motions for which the cross-shore mean Lagrangian velocity is zero. The $w_{x,y}(t)$ are zero-mean, stationary, white noise processes with variance (squared ensemble average) $\langle w_i(t) w_j(t') \rangle = \delta(t - t')\delta_{ij}$ so that $\int_0^t w_i(t') dt'$ are independent, incremental Weiner processes. Although the shoreline boundary condition for the observed drifters is mixed (due to terminating the trajectories of beached drifters), the model shoreline at $x = 0$ reflects particles resulting in a zero particle flux through the shoreline. The reflecting boundary condition is chosen as it is the appropriate boundary condition for tracer (dye, pollution) dispersion. Moreover, the results which follow are not significantly sensitive to the particular shoreline boundary condition (absorbing, reflecting, or mixed).

For $\tau_x = \tau_y = 0$, the set of four Langevin equations (4) reduces to two equations for the $(x,y)$ position with noise added directly to the positions [e.g., Zambianchi and Griffa, 1994], resulting in a classical Brownian random walk. The cross-shore equations (4a,c) do not depend on the alongshore variables $(y,v)$, thus cross-shore Lagrangian statistics are independent of the alongshore. On the other hand, alongshore Lagrangian statistics depend upon cross-shore Lagrangian model parameters due to $V(x)$.

3.2. LSM Analytic Solutions

Here, the ensemble-averaged analytic diffusivity is derived from the Langevin equations (4) for three situations. 1) Concepts and terminology are briefly introduced and reviewed for an unbounded (infinite) domain, before 2) considering a semi-infinite domain
corresponding to shoreline-bounded cross-shore surfzone drifter dispersion. Finally, 3) alongshore shear dispersion driven by cross-shore variable alongshore current $V(x)$ is considered. In contrast to the averaging over all available time-lags on a trajectory and over trajectories used in (2), analytic diffusivities $K(a)$ are derived using (ensemble) averages, represented here by $\langle \rangle$, which only average over trajectories.

### 3.2.1. Unbounded (Infinite) Domain

In an infinite domain without a mean current, the ensemble average statistics of (4a,c) are well known [see Uhlenbeck and Ornstein, 1930], and detailed derivations are omitted. Denoting a drifter trajectory as $x(t|x_0)$ with initial position $x_0$, i.e. $x(t = 0) = x_0$, and assuming a random initial velocity with mean square value $\sigma_u^2$, the ensemble mean position is

$$X(t|x_0) \equiv \langle x(t|x_0) \rangle = x_0$$

and the ensemble mean-squared position is

$$\overline{X^2}(t|x_0) \equiv \langle [x(t|x_0)]^2 \rangle = x_0^2 + 2\sigma_u^2 \tau_x [t + \tau_x e^{-t/\tau_x} - \tau_x]. \quad (5)$$

As positions are Gaussian [Uhlenbeck and Ornstein, 1930], the drifter positions pdf $P(x, t|x_0)$ at time $t$, for a particle released at $x_0$, is given by

$$P(x, t|x_0) = \frac{1}{\sqrt{2\pi \sigma_x^2(t)}} \exp \left[ -\frac{(x - x_0)^2}{2\sigma_x^2(t)} \right], \quad (6)$$

where the drifter displacement variance is defined as

$$\sigma_x^2(t) \equiv \overline{X^2}(t|x_0) - [X(t|x_0)]^2 = 2\kappa_x [t + \tau_x e^{-t/\tau_x} - \tau_x], \quad (7)$$
and $\kappa_x \equiv \sigma_x^2 \tau_x$ is the long-time diffusivity. The bulk diffusivity $K_x(t|x_0)$ is half the time-derivative of the drifter displacement variance,

$$K_x(t|x_0) \equiv \frac{1}{2} \frac{d}{dt} \sigma_x^2(t) = \kappa_x(1 - e^{-t/\tau_x}),$$  

recovering the classic ballistic ($K_x \approx \sigma_u^2 t$ for $t \ll \tau_x$) and Brownian ($K_x \approx \kappa_x$, for $t \gg \tau_x$) dispersion regimes [Taylor, 1922].

### 3.2.2. Bounded (Semi-Infinite) Domain

Here, the cross-shore diffusivity is derived for drifters released at $x_0$ ($x_0 < 0$) on a semi-infinite domain with a reflecting boundary at $x = 0$ and no mean cross-shore currents. This is considered the “half-line problem”. As the shoreline boundary reflects drifters, the boundary condition at $x = 0$ is no flux and the half-line (denoted by a subscript $H$) problem drifter position pdf $P_H(x, t, |x_0)$ is, using the method of images,

$$P_H(x, t| x_0) = P(x, t| x_0) + P(x, t| -x_0),$$  

where $P(x, t| x_0)$ is the infinite domain drifter position pdf (6). The half-line mean drifter position is

$$\bar{X}_H(t| x_0) = \int_{-\infty}^{0} x P_H(x, t| x_0) \, dx$$  

and performing the integral (10) gives a time-dependent mean position $\bar{X}_H(t|x_0)$ for the half-line problem,

$$\bar{X}_H(t|x_0) = -\sqrt{\frac{2}{\pi}} \sigma_x \exp \left[ -\frac{x_0^2}{2\sigma_x^2(t)} \right] - x_0 \text{erf} \left[ \frac{x_0}{2^{1/2} \sigma_x(t)} \right],$$  

where $\sigma_x^2(t)$ is given by (7). The half-line mean position (11) monotonically decreases with time as particles, unable to cross the shoreline, must eventually move negatively offshore.
The time when the shoreline is encountered, $t_0$, occurs when $\sigma_x^2(t_0) = x_0^2$. Defining the non-dimensional distance to the shoreline as

$$\alpha = \frac{x_0^2}{\kappa_x \tau_x},$$

(12)

the time $t_0$ is given by

$$\frac{t_0}{\tau_x} = \begin{cases} \alpha^{1/2} & \text{for } \alpha \ll 1, \\ \alpha/2 & \text{for } \alpha \gg 1. \end{cases}$$

(13)

These cases correspond to whether dispersion is ballistic (first case, $\sigma_x^2 = \sigma_u^2 t^2$) or Brownian (second case, $\sigma_x^2 = 2\kappa t$) when the boundary is encountered. For $t \ll t_0$ the mean drifter position is the release location and $X_H(t) \approx x_0$. For $t \gg t_0$ the mean drifter position is given by

$$X_H(t|x_0) \approx -\sqrt{\frac{2}{\pi}} \sigma_x(t).$$

(14)

The “half-line” mean-squared position

$$X_H^2 = x_0^2 + \sigma_x^2$$

is identical to that for the unbounded domain, but due to the mean position $X_H(t|x_0)$
time-dependence, the half-line bulk diffusivity $K_x^{(a)}(t|x_0)$

$$K_x^{(a)}(t|x_0) = \frac{1}{2} \frac{d}{dt} X_H^2(t|x_0) - X_H(t|x_0) \frac{d}{dt} X_H(t|x_0)$$

$$= \kappa_x (1 - e^{-t/\tau_x}) - X_H(t|x_0) \frac{d}{dt} X_H(t|x_0),$$

(15)

differs from the unbounded domain case. Plugging (11) into (15) yields,

$$K_x^{(a)}(t|x_0) = \kappa_x \left[ 1 - e^{-t/\tau_x} \right]\left[ 1 - \frac{2}{\pi} e^{-\alpha/2t''} - \sqrt{\frac{\alpha}{\pi t''}} e^{-\alpha/4t''} \text{erf} \left( \sqrt{\frac{\alpha}{4t''}} \right) \right]$$

(16)

where

$$t'' = t/\tau_x + e^{-t/\tau_x} - 1.$$
The presence of the shoreline boundary can significantly effect the bulk diffusivity. The half-line $K_x^{(a)} (16)$ is always less than that for the unbounded case (8). For drifters released at $x_0 = 0$, the boundary is instantly felt, and the bulk diffusivity is $K_x^{(a)} = \kappa_x (1 - 2/\pi) [1 - \exp(-t/\tau_x)]$ (see $\alpha = 0.001$ curve in Figure 3). For $x_0 \neq 0$ and $t \ll t_0$ the bulk diffusivity $K_x^{(a)} \approx \kappa_x [1 - \exp(-t/\tau_x)]$ tracks that for the infinite domain (see $\alpha = 10^3$ curve for $t/\tau_x < 10^2$ in Figure 3). For $\alpha = 10^3$ and for $t \gg t_0$, the bulk diffusivity tracks the half-line bulk diffusivity for particles released from $x_0 = 0$. The transition between these unbounded domain and shoreline-release behaviors occurs when $t \approx t_0$ resulting in a distinct diffusivity maxima for some $\alpha$ (see $\alpha = 10$ curve in Figure 3). For all $\alpha$, when $t \gg t_0$, the long-time bulk diffusivity is reduced by a factor of $(1 - 2/\pi)$ relative to the unbounded domain $K_x$.

For any pdf of initial release locations $p_0(x_0)$, the mean drifter location is calculated via,

$$X_H(t|p_0) = \int_{-\infty}^{0} p_0(x_0)X_H(t|x_0) \, dx_0$$

where $X_H(t|x_0)$ is given by (11). Note that $p_0(x_0)$ is distinct from the pdf of all cross-shore positions $p(x)$ (Figure 2a). Regardless of the initial release location pdf $p_0(x_0)$, the mean squared drifter position contribution to the diffusivity is $\kappa_x (1 - e^{-t/\tau_x})$. Therefore, the analytic diffusivity for any $p_0(x_0)$ is $K_x^{(a)}(t|p_0)$ and given by (15) with $X_H(t|p_0)$ instead of $X_H(t|x_0)$. The analytic cross-shore diffusivity $K_x^{(a)}$ will be compared to observed diffusivities in Section 4.1.

### 3.2.3. Alongshore Shear Dispersion

Here, analytic expressions for the alongshore drifter diffusivity induced by the sheared mean alongshore current $V(x)$ (i.e., shear dispersion) are presented. The theory of shear...
dispersion, first developed for laminar pipe flows [Taylor, 1953], was extended by Spydell and Feddersen [2011] to include the effect of non-zero cross-shore Lagrangian time-scale $\tau_x$. The alongshore diffusivity can be written as

$$K_y(t) = \kappa_y(1 - e^{-t/\tau_y}) + K_S(t), \quad (17)$$

where the first term on the rhs of (17) is the contribution of random alongshore particle motions due to surfzone eddies and the second term $K_S$ is the shear-induced diffusivity.

For an alongshore current $V(x)$ over cross-shore extent $L_V$, with Fourier cosine coefficients

$$V_n = \frac{2}{L_V} \int_0^{L_V} \cos(n\pi x/L_V)V(x) \, dx,$$

the analytic shear-induced bulk alongshore diffusivity $K_S(a)$ for cross-shore uniformly released drifters is [Spydell and Feddersen, 2011]

$$K_S(a)(t) = \sum_{n=1}^{\infty} \frac{V_n^2}{V^2} \int_0^t e^{-\frac{1}{2} \left(\frac{n\pi}{L_V}\right)^2 \sigma_x^2(t')} \, dt' \quad (18)$$

where $\sigma_x^2(t') = 2\kappa_x(t' + \tau_x e^{-t'/\tau_x} - \tau_x)$. The shear dispersion contribution to the observed alongshore diffusivity is analyzed using (18) in Section 4.2.

### 3.3. LSM Simulations of HB06 Data

In addition to comparing observed diffusivities to analytic ones, the HB06 drifter data set is stochastically reproduced using the LSM. For each day, modeled drifter trajectories are computed by numerically integrating the Langevin equations (4) using specified model parameters. A realization of modeled trajectories is obtained using the identical trajectory number, initial position (release location) and trajectory length, and with Gaussian-random initial velocity. From the simulated trajectories, modeled diffusivities $K(m)_{x,y}$ are calculated via (3) similar to the observed $K(o)_{x,y}$.
Unlike the analytic diffusivity solutions which are exact for a set of model parameters, the sampling error for diffusivity derived from trajectories (either observed or modeled) depends on the number of trajectories. For a single realization of a days worth of drifter observations, the observed diffusivity sampling error ($\epsilon_{K_x}$) can be significant [Spydell et al., 2009]. The diffusivity (or other moments) sampling error is approximately inversely proportional to the square-root of the number of drifters, thus the model diffusivity sampling error can be reduced to some predetermined threshold by increasing the number of simulated drifters. Here, ten times the number of observed drifters (10 realizations) are used in the model, reducing the modeled diffusivity sampling error ($\approx \epsilon_{K_x}/\sqrt{10}$).

For each day, the observed binned $V(x)$ (Figure 2b, where the cross-shore bins are 16–25 m wide depending on the day), with offshore $V(x) = 0$ (at $x = -1000$ m) is linearly interpolated to the drifter cross-shore position for use in the model. Although the cross- and alongshore drifter velocity variances vary in the cross-shore (Figure 2c,d), for simplicity bulk (scalar) cross- ($\sigma_u^2$) and alongshore ($\sigma_v^2$) velocity variances are used as model input. The $\sigma_u^2$ and $\sigma_v^2$ are calculated as the weighted (by drifter cross-shore position $p(x)$, Figure 2a) surfzone average of the observed cross-shore dependent $u^2(x)$ and $v^2(x)$ (Figure 2c,d), i.e., for $\sigma_u^2$,

$$\sigma_u^2 = \int_{x=-160 m}^{x=0 m} u^2(x) p(x) \, dx.$$  \hspace{1cm} (19)

Over the 5 release days, the bulk $\sigma_u^2$ varies between 0.014–0.025 m$^2$s$^{-2}$ (Table 2) and $\sigma_v^2$ varies between 0.020–0.028 m$^2$s$^{-2}$ (Table 3).

Unlike drifter velocity variances, the spatially-constant cross- ($\tau_x$) and alongshore ($\tau_y$) Lagrangian time-scales cannot be a priori derived from the drifter observations without in-
voking additional assumptions. In the open ocean, the Lagrangian time-scale is typically between 0.5–1.25 times the Eulerian time-scale [Lumpkin et al., 2002] and is consistent with the Middleton [1985] formula. Lagrangian time-scales smaller than Eulerian time-scales are expected for eddies that are relatively strong, small, and persistent. Although the precise relationship between Lagrangian and Eulerian time-scales is unknown in the surfzone, as the horizontal eddies responsible for dispersion have significant energy in the Eulerian time-scales of 33–250 s [Feddersen et al., 2011], \( \tau_x \) and \( \tau_y \) are expected to fall within this range. Within this \( \tau_{x,y} \) range, modeled and observed drifter trajectories are qualitatively similar (Figure 4).

The modeled diffusivities \( K_x^{(m)}(t) \) depend on \( \tau_x \) (Figure 5). For example, on release day 09/17, the modeled \( K_x^{(m)} \) (i.e., the mean of the diffusivities found for the 10 realizations of modeled 09/17 drifter tracks) with \( \tau_x = 125 \) s is very similar to the observed diffusivity \( K_x^{(o)} \) over the entire 1000 s of observations (compare solid and dashed curves in Figure 5). The uncertainty in the modeled diffusivity is small and is approximately \( \sim \pm \epsilon_{K_x}/10^{1/2} \) (\( \epsilon_{K_x} \) is the gray shading in Figure 5). For smaller \( \tau_x = 50 \) s, \( K_x^{(m)} \) deviates significantly from the observed \( K_x^{(o)} \) by ramping up too quickly to half the observed maximum and no long-time decay (thin solid curve in Figure 5). Similarly, the \( \tau_x = 200 \) s modeled \( K_x^{(m)} \) overshoots the observations, with a 50% larger maximum occurring 50% later than observed and decaying more rapidly at long times (dash-dot curve in Figure 5).

On each release day, the best-fit cross-shore Lagrangian time-scale \( \tau_x \) is chosen to minimize the squared difference between the modeled and observed bulk diffusivity \( s^2(\tau_x) \), i.e.,

\[
 s^2_{K_x}(\tau_x) = \frac{1}{t_{\text{max}}} \int_0^{t_{\text{max}}} \left[ K_x^{(o)}(t) - K_x^{(m)}(t; \tau_x) \right]^2 dt ,
\]  

(20)
where \( t_{\text{max}} = 1000 \) s. For each release day, the Langevin equation model (4) is run for various \( \tau_x \) spanning 30–300 s (Section 3.3). For all release days, the best-fit \( \tau_x \) varies between 75–200 s (Table 2), qualitatively consistent with the 116–190 s observed cross-shore diffusivity time-scale \([\text{Spydell et al.}, 2009]\) and consistent with the Eulerian time-scale (inverse frequency) of surfzone eddies \([\text{Feddersen et al.}, 2011]\). Qualitative (non-statistical) \( \tau_x \) “error bars” are defined as the \( \tau_x \) that increases \( s_{K_x} \) by 20\% of \( \langle (K_x^{(o)})^2 \rangle^{1/2} \), where \( \langle \rangle \) represents a time-average over 1000 s. The \( \tau_x \) error-bars range between 16–69 s on the five release days (Table 2). The best-fit \( \tau_y \) (and error-bars) are found by running the model with the best-fit \( \tau_x \) and over a 30–300 s \( \tau_y \) range and minimizing the mean-square error \( s_{K_y}^2(\tau_y) \) defined analogous to (20). The resulting \( \tau_y \) ranges between 95–155 s, similar to the best-fit \( \tau_x \) range (Table 3) with \( \tau_y \) error-bars about twice the \( \tau_x \) error bars.

4. Results

In this section, bulk cross- and alongshore diffusivities from the HB06 observations \( K^{(o)} \), LSM modeled \( K^{(m)} \), and analytic solution \( K^{(a)} \) are compared. The analytic diffusivity \( K_x^{(a)}(t) \) (see Section 3.2.2) is calculated on each release day using \( \sigma_u^2 \) defined in (19), the best-fit \( \tau_x \), and the daily pdf of cross-shore drifter initial release locations \( p_0(x_0) \).

4.1. Cross-shore Diffusivity \( K_x \)

For all release days (except 10/14), the modeled \( K_x^{(m)} \) and the analytic \( K_x^{(a)} \) reproduce the observed \( K_x^{(o)} \) (Figure 6) with small root-mean-square errors (rmse) and high skill (Table 2) using the best-fit \( \tau_x \). The modeled cross-shore diffusivity time-dependence is due to the shoreline, and strongly suggests that the non-monotonic \( K_x^{(o)}(t) \) is also due to the shoreline. Focusing on 09/17, the model \( K_x^{(m)} \) accurately reproduces the ballistic
regime \((t < 50 \text{ s})\), the \(K_x^{(o)}\) maximum at \(t \approx 310 \text{ s}\), and the slow \(K_x^{(o)}\) decrease for longer \(t\) (Figure 6a1,a2). For all release days, \(K_x^{(m)}\) and \(K_x^{(a)}\) reproduce well \(K_x^{(o)}\) (compare the dashed, solid, and dash-dot curves in Figure 6, left column) the short time \((t < 50 \text{ s and } t/\tau_x < 0.7)\) ballistic regime \(K_x = \sigma_u^2 t\). With the exception of 10/14, the modeled \(K_x^{(m)}\) reproduces the observed \(K_x^{(o)}\) at longer times \((t > 400 \text{ s or } t/\tau_x > 3)\), left and right columns of Figure 6, respectively). Similarly, at longer times, the analytic \(K_x^{(a)}\) also reproduces \(K_x^{(o)}\) on 10/02 and 10/03, but somewhat underpredicts \(K_x^{(o)}\) on 09/17 and 10/15 with slightly smaller skills. At intermediate times, there is variation in how well the model and analytic \(K_x\) represent the observed \(K_x^{(o)}\). On 09/17 and 10/15, the modeled \(K_x^{(m)}\) reproduces well the magnitude and timing of the maximum \(K_x^{(o)}\) (Figure 6a1,a2,e1,e2) whereas \(K_x^{(a)}\) is generally smaller for \(t > 200 \text{ s or } t/\tau_x > 1.5\). On 10/02 and 10/03, the modeled and analytic \(K_x\) underpredict the \(K_x^{(o)}\) maximum by approximately 1/3 but reproduce the timing of the maximum (Figure 6b1,b2,c1,c2).

Although \(K_x^{(m)}\) and \(K_x^{(a)}\) are statistics of the same LSM (4) with identical release locations, \(K_x^{(a)}\) and \(K_x^{(m)}\) differ due to the differences in averaging. The observed diffusivities \(K_x^{(o)}\) are generally limited to maximum times of \(4\tau_x\) to \(10\tau_x\) (Figure 6, right column), due to the limited duration of drifter trajectories. Thus, the long-time limit of \(K_x^{(a)}/\kappa_x = 1 - 2/\pi\) (dashed line on Figure 6 right column) is generally not observed. The average cross-shore release location \(\bar{x}_0\) varies between \(-38\) and \(-73 \text{ m}\) resulting in a bulk \(\alpha\) between 3–40, indicating that the shoreline is encountered between the ballistic and Brownian regimes. The non-dimensional \(K_x/\kappa_x\) dependence upon \(t/\tau_x\) (Figure 6a2-e2) is qualitatively similar to the point release analytic solution for \(\alpha = 10\) (Figure 3).
On 10/14, the model and analytic diffusivities poorly represent the observed diffusivity at both intermediate times \((t/\tau_x \approx 1)\) and the rapid decay at long times (Figure 6d1,d2) with highest rms error \((0.55 \text{ m}^2\text{s}^{-1})\) and the lowest skill of 0.87 (Table 2). As discussed in Spydell et al. [2009], at longer times drifter trajectories converged in the inner-mid surfzone suggesting bathymetric control induces this long-time rapid decay.

### 4.2. Alongshore Diffusivity \(K_y\)

On all release days, the model \(K_y^{(m)}\) reproduces well the observed \(K_y^{(o)}\) for the entire 1000 s (Figure 7, compare dashed and solid curves) with small rms errors and high skill \((>0.99, \text{ Table 3})\). For all times \((\leq 1000 \text{ s})\), the model \(K_y^{(m)}\) is within the observed \(K_y^{(o)}\) uncertainty (gray shaded region in Figure 7) on all release days. The best-fit \(\tau_y\) (Table 3) are in the same range as the best-fit \(\tau_x\) (Table 2). The \(\tau_y\) error bars are relatively broad indicating that a wide \(\tau_y\) range would give accurate results.

The relative contributions of the unbounded dispersion \((i.e., \text{ the first term on rhs of Eq. 17, denoted by } K_y^{(V=0)})\) and the shear-induced \(K_S\) (the 2nd term in Eq. 17) is not understood. The shear-induced component of \(K_y\) is isolated using the \(V = 0\) analytic unbounded diffusivity solution \(K_y^{(V=0)} = \kappa_y[1 - \exp(-t/\tau_y)]\) (thin curves in Figure 7). On all release days at times \(\leq \tau_y\), the \(K_y^{(V=0)}\) matches \(K_y^{(o)}\), indicating that at short times the diffusion is dominated by unbounded random motions (thin curves in Figure 7). On the three strong \(V(x)\) release days (09/17, 10/14, and 10/15 in Figure 2b), \(K_y^{(V=0)}\) has a much shorter ramp-up time-scale and a much smaller long-time diffusivity than \(K_y^{(o)}\) (compare thin and dashed curves in Figure 7a,d,e). In contrast, on the 2 days (10/02 and 10/03) with weak \(V(x)\) and weak shear, the \(K_y^{(V=0)}\) was only 20–30% less than the observed
suggesting that shear dispersion $K_S$ was less important than surfzone eddy-induced random dispersion.

The shear-induced diffusivity $K_S$ contribution is quantified for the observations and model by using

$$K_{S}^{(o,m)}(t) = K_{y}^{(o,m)}(t) - K_{y}^{(V=0)}.$$  (21)

The analytic shear-induced diffusivity $K_{S}^{(a)}$ is also calculated using (18) with the Fourier coefficients of the observed $V(x)$, the model $\kappa_x$ and best-fit $\tau_x$, and the drifter-observed alongshore current $L_V$ width (essentially 160 m, Table 4). The observed $K_{y}^{(o)}$, modeled $K_{y}^{(m)}$, and analytic diffusivities $K_{S}^{(a)}$ are compared in Figure 8 and Table 4.

The observed $K_{S}^{(o)}$ and modeled $K_{S}^{(m)}$ shear-induced diffusivity is well reproduced by the analytic $K_{S}^{(a)}$ on all release days (Figure 8) with high skill ($\geq 0.95$ except for 10/03 where skill was 0.92, Table 4). The shear-dispersion dominated release days (09/17, 10/14, and 10/15), have large $K_S (> 7 \text{ m}^2\text{s}^{-1})$ and have a long ramp-up time-scale (Figure 8a,d,e). The difference between the observed $K_{S}^{(o)}$ and analytic $K_{S}^{(a)}$ is within the uncertainty of the observed $K_{y}^{(o)}$ (gray shaded region in Figure 7). This indicates that the analytic solution of Spydell and Feddersen [2011] for uniformly distributed cross-shore release is applicable for both low and strong current shear.

For the strong $V(x)$ release days, the alongshore diffusivity $K_y$ is dominated by shear dispersion. Consistent with this, LSM simulations with the observed $V(x)$ and small $\tau_y = 10 \text{ s}$ yield $K_{y}^{(m)}$ that are similar to $K_{y}^{(o)}$ (not shown) with relatively small errors and high skill ($> 0.91$). Even on the weak $V(x)$ release days (10/02 and 10/03) the shear-induced $K_S$ is greater than or equal to the cross-shore diffusivity $K_{x}^{(o)}$ (Figure 6b1,c1), indicating that for alongshore uniform beaches, shear-induced dispersion is as, or more
important than, cross-shore dispersion (i.e., $K_x$) in surfzone drifter or tracer dispersion (and thereby dilution).

5. Discussion

5.1. Cross-shore Diffusivity

A Lagrangian stochastic model, that includes the shoreline and has cross-shore uniform velocity variance $\sigma_u^2$ and Lagrangian time-scale $\tau_x$, reproduces the first 1000 s of the non-monotonic observed cross-shore diffusivity $K_x^{(o)}$ (Figure 6). This includes the ballistic regime, the magnitude and timing of the $K_x^{(o)}$ maxima, and the longer time decay. On 3 of the 5 days the cross-shore diffusivity maximum is under predicted by about 30% and may be due to the use of cross-shore constant velocity variance and Lagrangian time-scale when they are cross-shore dependent. The modeled and analytic $K_x^{(m,a)}$ maximum and longer-time decay is due to the presence of the shoreline strongly suggesting that the observed non-monotonic $K_x^{(o)}$ is also due to the shoreline. The hypothesis that diffusivity is weaker seaward of the surfzone (either by smaller $\sigma_u^2$ or $\tau_x$) is not necessary to explain the first 1000 s of the observed $K_x$ time-dependence. The model has high skill with bulk cross-shore uniform $\sigma_u^2$ and $\tau_x$, even though the observed cross-shore velocity variance $\overline{u^2}$ is clearly cross-shore dependent (Figure 2c) and the cross-shore dependence of $\tau_x$ is not known. This suggests that, for at least the first 1000 s, the effect of cross-shore variable statistics is weak.

Assuming that a region of reduced diffusivity seaward of the surfzone exists (i.e., $\kappa_x^{(sea)} < \kappa_x^{(surf)}$), then the diffusive time $\tau_D = L^2 / \kappa_x$ (where $L$ is the cross-shore length-scale of the surfzone) required for surfzone released particles to begin to “feel” the reduced diffusivity is $\approx 10,000$ s (Table 4), much longer than the 1000 s of observed
dispersion. At times greater than $\tau_D$, the bulk drifter-derived cross-shore diffusivity would be reduced by the lower seaward-of-the-surfzone diffusivity region and approach $(1 - 2/\pi)\kappa_x^{(sea)}$, where the factor $(1 - 2/\pi)$ is due to the shoreline. If horizontal diffusion is significantly weaker seaward of the surfzone, the HB06 drifter tracks were too short to observe it.

For the modeled and analytic cross-shore diffusivities, particles are reflected at the shoreline resulting in a no flux boundary condition. However, the GPS-tracked drifters did not follow this boundary condition. Drifters that came too close to the shoreline and were grounded (or “beached”) had their trajectories terminated and were re-released further offshore. Thus, the boundary condition for the observed drifters is mixed: some drifters reflected off the shoreline while others (the ones which grounded) were absorbed. Repeating the analysis that led to (15), but with an absorbing (rather than no-flux) boundary condition, results in an analytic cross-shore diffusivity similar to the no-flux $K_x^{(a)}$. For $\alpha \approx 10$ (the order of magnitude here, Table 2), the absorbing diffusivity also has a distinct maximum and at long time approaches a constant $\kappa_x(2 - \pi/2) = 0.43\kappa_x$ which is only slightly larger than the no-flux constant $\kappa_x(1 - 2/\pi) = 0.36\kappa_x$.

### 5.2. Alongshore Diffusivity

The analytic shear-induced diffusivity $K_S^{(a)}$ accurately reproduces the observed shear-induced alongshore dispersion (Figure 8) for both strong and weak alongshore current shear release days. Although the observed drifter position pdf is not uniform and varies from day to day (Figure 2a), the cross-shore uniform drifter distribution assumption used in (18) does not affect the results. In Spydell et al. [2009], the alongshore diffusive time-scale $T_y$ was found by (essentially) fitting the observed alongshore diffusivity $K_y^{(o)}(t)$ to

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\(\kappa_y[1 - \exp(-t/T_y)]\). For the 3 days with strong shear dispersion (09/17, 10/14, and 10/15) this \(T_y\) varied between 340–420 s (Table 1 in Spydell et al. [2009]), significantly larger than the best-fit Lagrangian time-scale \(\tau_y\) found here (Table 3). In contrast, when shear dispersion was weak and the alongshore diffusivity was dominated by unbounded random dispersion (10/02 and 10/03), the Spydell et al. [2009] Lagrangian time-scale \(T_y = (190, 118)\) s is consistent with the best-fit \(\tau_y = (140, 100)\) s here.

Given the \(L_V, \kappa_x,\) and \(\tau_x\) parameters for the strong \(V(x)\) days (see Tables 2 and 4), \(\tau_D = L_V^2/\kappa_x\) varies between 8000–10,700 s and \(\tau_x/\tau_D\) varies between 0.012–0.025 (Table 4).

For \(\tau_S/\tau_D \approx 0.01\), the shear-dispersion \(K^{(a)}_S(t)\) time-dependence straddles the very small \(\tau_x\) \((\ll \tau_D)\) exponential \((\propto 1 - \exp(-\pi^2 t/\tau_D))\), and the moderate \(\tau_x\) error-function \((\propto \text{erf}(t/\tau_S))\), where \(\tau_S = (2\tau_x\tau_D)^{1/2}/\pi\) is the appropriate shear dispersion time-scale [Spydell and Feddersen, 2011] time-dependence. Although \(K^{(a)}_S(t)\) is neither of these functions, the time-scale \(\tau_S\) varies between 480–580 s over all release days, consistent with the ramp-up time for the observed, modeled, and analytic \(K_S(t)\) (Figure 8). Moreover, on the 3 strong \(V(x)\) release days, \(\tau_S\) is also qualitatively consistent with the observed Spydell et al. [2009] \(K^{(o)}_y(t)\) time-scale \(T_y\) (340–420 s) that combines random unbounded dispersion and shear-dispersion. This further demonstrates the consistency of the non-zero \(\tau_x\) shear-dispersion theory, and also demonstrates that the ramp-up time-scale of the alongshore diffusivity cannot be interpreted as a Lagrangian time-scale.

The reason Spydell et al. [2009] inferred a factor of 3 difference from shear-dispersion theory was because a parabolic \(V(x)\) profile was used in order to directly apply the classic Taylor [1953] theory for laminar \(\tau_x = 0\) shear-dispersion. Furthermore, the shear-dispersion theory of Spydell and Feddersen [2011], used here to estimate \(K^{(a)}_S\), is applicable
for any Lagrangian time-scale $\tau_x \geq 0$. For the values of $\tau_x/\tau_D$ observed on the strong $V(x)$
release days (0.012–0.025, Table 4), the classic Taylor theory using the observed $V(x)$
yields shear-induced diffusivities that are 10-20% smaller than the $K^{(a)}_S$ calculated here.
Thus, the Taylor [1953] theory can underpredict shear-induced dispersion in the surfzone
or in other turbulent flows where the cross-domain Lagrangian time-scale is sufficiently
large.

5.3. Qualitatively simulating pollution dispersal: LSM model parameters

Complex wave-resolving or wave-averaged surfzone wave and circulation models with
a coupled tracer or drifter model can be used to simulate the transport and dispersal
of a beach (shoreline) pollution spill. Such models have been used to reproduce surfzone
observed drifter dispersion [Spydell and Feddersen, 2009; Reniers et al., 2009] and surfzone
observed dye tracer transport and dispersion [Clark et al., 2011]. Such models contain
the detailed physics of surfzone eddy-induced stirring [e.g., Spydell and Feddersen, 2009;
Long and Özkan-Haller, 2009; Reniers et al., 2009] that induces horizontal dispersion
of drifters and tracer [Spydell et al., 2007; Spydell and Feddersen, 2009; Clark et al.,
2010, 2011; Geiman et al., 2011]. However, the complexity of these models also can be
prohibitive in terms of setting them up and running them sufficiently rapidly to provide
useful qualitative estimates of surfzone pollution dispersal.

Given the model parameters $V(x), \sigma_u^2, \sigma_v^2, \tau_x$, and $\tau_y$, the LSM model can be used both
numerically and analytically to predict well at least the first 1000 s of the transport and
dispersion of pollution (e.g., shoreline contaminant spill). Here, the model parameters
are derived from the drifter observations. However, in general LSM model parameters
need to be apriori specified to simulate pollution dispersion. These parameters can be
approximately estimated with varying levels of uncertainty from relatively simple models
(e.g., \(V(x)\)), empirical relationships (e.g., \(\sigma_u^2\)), and from knowledge of surfzone eddy
Eulerian time-scales (\(\tau_x\), and \(\tau_y\)).

Shear-dispersion is often the most significant component to the overall diffusivity, par-
ticularly for significant alongshore current shear (as on 09/17, 10/14, and 10/15). Thus,
resolving the mean alongshore current \(V(x)\) is critical. Both very simple one-dimensional
alongshore current models [e.g., Ruessink et al., 2001] or more complex wave-averaged
[e.g., Shi et al., 2011] or wave-resolving [e.g., Chen et al., 2003; Feddersen et al., 2011]
models have been shown to model \(V(x)\) well given the bathymetry and incident wave
field.

The bulk (cross-shore averaged) low-frequency \((f < 0.03 \text{ Hz})\) Lagrangian velocity vari-
ances \(\sigma_u^2\) and \(\sigma_v^2\) are derived from the low-frequency quasi-Eulerian cross-shore dependent
\(\overline{u^2}(x)\) and \(\overline{v^2}(x)\). For velocities which include sea-swell motions as well as low-frequency
ones, quasi-Eulerian binned drifter velocity variances are similar to ADV-measured Eu-
lerian velocity variances [see Figure 2b,c in Spydell et al., 2009]. Similarly, the low-
frequency drifter-derived \(\overline{u^2}(x)\) and \(\overline{v^2}(x)\) are correlated with \((r = 0.72\) and \(r = 0.67)\)
but are roughly 1.3 and 1.9, respectively, times larger than ADV-measured low-frequency
velocity variances (Figure 9). An Eulerian bulk rotational velocity variance [Lippmann
et al., 1999], that removes the infragravity wave component but combines cross- and along-
shore velocities, is similar to the Eulerian low-frequency velocity variance (not shown).
Thus, at low-frequencies, infragravity wave motions, which do not lead to diffusion, are
weak relative to rotational velocities. The relationship between the binned-drifter and
ADV-measured velocity variances (Figure 9) implies that knowledge of the Eulerian low-
frequency (< 0.03 Hz) velocity variance can be used to approximately estimate the \( \sigma_u^2 \) and \( \sigma_v^2 \), albeit with substantial uncertainty.

Without in situ observations, the Eulerian low-frequency velocity variance must still be specified. Although the Eulerian low-frequency velocities can be reasonably well modeled with a complex surfzone circulation model [Reniers et al., 2007; Feddersen et al., 2011], running such a model goes against the idea of using a simple LSM. Eulerian rotational velocities (responsible for the dispersion) are driven by both shear-instabilities [Allen et al., 1996] and wave-breaking of a directionally spread wave field [Peregrine, 1998; Spydell and Feddersen, 2009]. Although the contribution of each driver is not well understood, bulk (low-frequency) rotational velocities are approximately linearly related to the local mean alongshore current magnitude \( |V| \) (see Figure 8 in Noyes et al. [2004] and also Figures 14,15 in Feddersen et al. [2011]). The bulk rotational velocities are roughly evenly split between \( \bar{u}^2 \) and \( \bar{v}^2 \) [e.g., Noyes et al., 2004], except near the shoreline where \( \bar{v}^2 \) dominates [Feddersen et al., 2011]. These observations can provide a simple way to approximately parameterize \( \sigma_u^2 \) and \( \sigma_v^2 \) using \( V(x) \).

The Lagrangian time-scale remains to be specified, and has the largest uncertainty in apriori specification. Here, the best-fit \( \tau_x \) ranged between 75–200 s, and observed drifter-derived \( K_x^{(o)} \) ramp-up times are always less than 200 s [Spydell et al., 2007, 2009; Brown et al., 2009]. The best-fit Lagrangian time-scale \( \tau_x \) is not correlated with other potential time-scales (e.g., peak wave period or \( L/\sigma_u \)) nor wave/current conditions (e.g., \( H_s \) or \( V \)). However, the variation of these parameters may have been too small to observe any such potential relationship.
The relationship between Eulerian ($\tau_E$) and Lagrangian time-scales in the surfzone is not understood. However, in the open ocean the ratio of Lagrangian to Eulerian $\tau$ generally ranges between 0.5–1.25 [e.g., Lumpkin et al., 2002]. Using the Lagrangian time-scale range of $\tau$ found here and inferred Eulerian time-scales of surfzone eddies [e.g., Noyes et al., 2004; Spydell and Feddersen, 2009; Long and Özkan-Haller, 2009], the ratio $\tau/\tau_E$ in the surfzone is similar to the open-ocean observations. Generally, weaker $V(x)$ release days have significantly (relative to the error-bars) smaller best-fit $\tau_x = 75$ s than stronger $V(x)$ release days with $\tau_x \geq 125$ s (Table 2). This may reflect reduced longer-timescale shear-instability induced eddies and increased shorter time-scale directionally-spread breaking-wave generated eddies relative to stronger $V(x)$ release days. The best-fit $\tau_x$ and $\tau_y$ are generally similar (Tables 2 and 3) and for strong $V(x)$, where shear-induced alongshore dispersion dominates, varying $\tau_y$ has a small effect on $K_y$ at longer-times. Therefore, although significant uncertainty remains, setting $\tau_x = 75$ s for weak $V(x)$ and larger $\tau_x$ ($\approx 150$ s) for stronger $V(x)$, together with $\tau_y = \tau_x$, may give qualitatively reasonable diffusivity estimates. One benefit of the analytic solutions, is that the $K$ variation induced by varying the Lagrangian time-scale can be readily evaluated.

6. Summary

The observed cross- ($K_x$) and alongshore ($K_y$) diffusivities for 5 release days on an alongshore uniform beach were reproduced by Lagrangian stochastic model (LSM) simulations and analytic solutions. The model solves for particle trajectories using the observed bulk Lagrangian velocity variance and alongshore current. The best-fit cross- and alongshore Lagrangian time-scales, found by minimizing the difference between the modeled and observed diffusivities, generally range between 75–200 s, and are consistent with the
Eulerian time-scale (inverse frequency) of surfzone eddies. Although velocity variances are considerably cross-shore dependent, the model works well assuming homogeneous velocity variances and Lagrangian time-scales.

The features of the time-dependent observed $K_x$, initial ballistic growth to a maximum and slow long-time decay, are explained by the presence of a shoreline boundary. Weaker diffusivity seaward of the surfzone is not required to explain the observed $K_x$ features over 1000 s. The observed drifter trajectories were too short to observe the analytic long-time limit for $K_x$ or to deduce whether the seaward of the surfzone diffusivity is weaker than within the surfzone.

The alongshore diffusivity $K_y$ has two components: a random unbounded dispersion component due to surfzone eddies and a shear-dispersion component $K_S$ induced by cross-shore shear in the alongshore current $V(x)$. On release days with moderate-to-strong $V(x)$, the alongshore diffusivity was dominated by shear-dispersion $K_S$. Even on weak $V(x)$ release days where shear-dispersion $K_S$ is about 1/3 of $K_y$, shear-dispersion $K_S$ is approximately the same magnitude of the cross-shore diffusivity $K_x$. The analytic $K_S$, which assumes cross-shore uniformly distributed drifters and includes the effects of non-zero Lagrangian time-scale, accurately reproduces the inferred shear-dispersion on all release days (both strong and weak $V(x)$). Although not in the correct asymptotic regime, an approximate analytic time-scale is qualitatively consistent with the observed $K_S$ time-scale.

With apriori knowledge of the bathymetry and incident wave field, the alongshore current $V(x)$ can be accurately predicted, and can be used to parameterize the Lagrangian
stochastic model inputs, allowing for simple predictions of the dispersal of an initial value problem of a beach contaminant spill.

Acknowledgments. This analysis was supported by NSF, ONR, and CA Sea Grant. The HB06 field work was supported by CA Coastal Conservancy, NOAA, NSF, ONR, and CA Sea Grant. R. T. Guza was a co-PI on the HB06 experiment. Staff and students from the Integrative Oceanography Division (B. Woodward, B. Boyd, K. Smith, D. Darnell, I. Nagy, D. Clark, M. Omand, M. Yates, M. McKenna, M. Rippy, S. Henderson) were instrumental in acquiring the field observations.

References


Haile, R. W., et al., The health effects of swimming in ocean water contaminated by storm

Jiang, S. C., and W. Chu, PCR detection of pathogenic viruses in Southern California

Lippmann, T. C., T. H. C. Herbers, and E. B. Thornton, Gravity and shear wave con-

Long, J. W., and H. T. Özkan-Haller, Low-frequency characteristics of wave group–forced

Lumpkin, R., A. Treguier, and K. Speer, Lagrangian eddy scales in the northern Atlantic


Noyes, T. J., R. T. Guza, S. Elgar, and T. H. C. Herbers, Field observations of shear

Pasquero, C., A. Provenzale, and A. Babiano, Parameterization of dispersion in two-
dimensional turbulence, *J. Fluid Mech.*, 439, 279–303, doi:10.1017/S0022112001004499,


Reeves, R. L., S. B. Grant, R. D. Mrse, C. M. C. Oancea, B. F. Sanders, and A. B.
Boehm, Scaling and management of fecal indicator bacteria in runoff from a coastal


Figure 1. The drifter-derived observed (a) cross-shore $K_x$, and (b) alongshore $K_y$ diffusivity versus time on the 5 HB06 release days given by the legend [from Spydell et al., 2009].

Table 1. Incident significant wave height $H_s$, maximum alongshore current $V_m$, and surfzone width $L_{sz}$ on the five drifter release days as reported in Spydell et al. [2009].

<table>
<thead>
<tr>
<th>day</th>
<th>09/17</th>
<th>10/02</th>
<th>10/03</th>
<th>10/14</th>
<th>10/15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_s$ [m]</td>
<td>0.83</td>
<td>0.68</td>
<td>0.65</td>
<td>0.69</td>
<td>0.68</td>
</tr>
<tr>
<td>$V_m$ [ms$^{-1}$]</td>
<td>0.27</td>
<td>-0.13</td>
<td>-0.17</td>
<td>0.35</td>
<td>0.25</td>
</tr>
<tr>
<td>$L_{sz}$ [m]</td>
<td>99</td>
<td>74</td>
<td>79</td>
<td>79</td>
<td>79</td>
</tr>
</tbody>
</table>
Figure 2. Binned drifter statistics versus the cross-shore coordinate $x$ on each 5 HB06 release day (colors): (a) the probability distribution function of cross-shore drifter position $p(x)$, (b) mean alongshore current $V(x)$, (c) low-frequency ($f < 0.03$ Hz) cross-shore $u^2(x)$ and (d) alongshore $v^2(x)$ drifter velocity variances. Cross-shore bin width varies between 16–25 m, and the cross-shore extent of Lagrangian observations varied between 150–169 m, depending on the day.
The analytic half-line non-dimensional diffusivity $K_x^{(a)}/\kappa_x$ versus non-dimensional time $t/\tau_x$ (16). Line thickness indicates various non-dimensional release location $\alpha = x_0^2/(\kappa_x \tau_x)$ (see legend).

Table 2. Various cross-shore diffusivity $K_x$ LSM parameters for the 5 HB06 drifter release days. The model (analytic) $K_x$ root-mean-square-error (rmse) is defined as the $\langle (K_x^{(m)} - K_x^{(o)})^2 \rangle^{1/2}$ and skill is defined as $1 - \text{rmse}^2/\langle (K_x^{(o)})^2 \rangle$, where $\langle \rangle$ denote a 1000 s time-average. The analytic $K_x^{(a)}$ rmse and skill are shown in parentheses. The $\tau_x$ error bars are defined as the $\tau_x$ change that increases the rmse by $0.2[\langle (K_x^{(o)})^2 \rangle]^{1/2}$. The $\bar{x}_0$ is the average over all release locations (open circles in Figure4) and $\alpha$ (12) is estimated with $x_0$.

<table>
<thead>
<tr>
<th>day</th>
<th>09/17</th>
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<th>10/03</th>
<th>10/14</th>
<th>10/15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_u^2$ [m$^2$s$^{-2}$]</td>
<td>0.017</td>
<td>0.023</td>
<td>0.025</td>
<td>0.018</td>
<td>0.014</td>
</tr>
<tr>
<td>$\tau_x$ [s]</td>
<td>125(±30)</td>
<td>75(±20)</td>
<td>75(±15)</td>
<td>125(±35)</td>
<td>200(±70)</td>
</tr>
<tr>
<td>$K_x^{(m)}$ ($K_x^{(a)}$) rmse [m$^2$s$^{-1}$]</td>
<td>0.06 (0.37)</td>
<td>0.22 (0.15)</td>
<td>0.25 (0.22)</td>
<td>0.55 (0.55)</td>
<td>0.16 (0.42)</td>
</tr>
<tr>
<td>$K_x^{(m)}$ ($K_x^{(a)}$) skill</td>
<td>0.99 (0.94)</td>
<td>0.97 (0.98)</td>
<td>0.95 (0.96)</td>
<td>0.87 (0.87)</td>
<td>0.99 (0.91)</td>
</tr>
<tr>
<td>$\bar{x}_0$ [m]</td>
<td>-73</td>
<td>-71</td>
<td>-58</td>
<td>-55</td>
<td>-38</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>20</td>
<td>40</td>
<td>23</td>
<td>11</td>
<td>3</td>
</tr>
</tbody>
</table>
Figure 4. The (a) observed and (b) LSM modeled drifter trajectories (thin black curves) on 09/17 with \((\tau_x, \tau_y) = (125, 95)\) s. Open white circles are drifter release locations and the thick grey curve near \(x = 0\) m is the approximate shoreline. The dashed grey line indicates the outer edge of the surfzone \((L_{sz}\) in Table 1). Bathymetry contours (thin gray) are shown at 1 m intervals.
Figure 5. The observed (dashed) and LSM modeled cross-shore diffusivity $K_x$ versus time $t$ for release day 09/17 for $\tau_x = (50, 125, 200)$ s (see legend). The gray shading indicates the observed sampling error [Spydell et al., 2009].

Table 3. Alongshore diffusivity $K_y$ LSM model parameters and skills for the 5 HB06 drifter release days. The $K_y$ root-mean-square-error (rmse) and skill and the $\tau_y$ error bars are defined as in (see Table 2).

<table>
<thead>
<tr>
<th>day</th>
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<th>10/15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_v^2$ [m$^2$s$^{-2}$]</td>
<td>0.020</td>
<td>0.027</td>
<td>0.028</td>
<td>0.025</td>
<td>0.020</td>
</tr>
<tr>
<td>$\tau_y$ [s]</td>
<td>95($\pm$65)</td>
<td>140($\pm$50)</td>
<td>100($\pm$30)</td>
<td>105($\pm$80)</td>
<td>155($\pm$65)</td>
</tr>
<tr>
<td>$K_y^{(m)}$ rmse [m$^2$s$^{-1}$]</td>
<td>0.43</td>
<td>0.13</td>
<td>0.21</td>
<td>1.02</td>
<td>0.30</td>
</tr>
<tr>
<td>$K_y^{(m)}$ skill</td>
<td>0.99</td>
<td>0.99</td>
<td>0.99</td>
<td>0.99</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Table 4. Alongshore shear-induced diffusivity $K_S$ parameters and model errors and skills for the 5 HB06 drifter release days. The cross-shore extent of the alongshore current is given as $L_V$. The $K_S$ root-mean-square-error (rmse) is defined as $\langle (K_S^{(a)} - K_S^{(o)})^2 \rangle$ and skill is defined as in Table 2. The ratio $\tau_x/\tau_D$ is calculated with the parameters in Table 2 and $\tau_D = L_V^2/\kappa_x$.

<table>
<thead>
<tr>
<th>day</th>
<th>09/17</th>
<th>10/02</th>
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<th>10/14</th>
<th>10/15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_V$ [m]</td>
<td>150</td>
<td>167</td>
<td>169</td>
<td>150</td>
<td>151</td>
</tr>
<tr>
<td>$K_S^{(a)}$ mse [m$^2$s$^{-1}$]</td>
<td>0.40</td>
<td>0.09</td>
<td>0.28</td>
<td>2.28</td>
<td>0.54</td>
</tr>
<tr>
<td>$K_S^{(a)}$ skill</td>
<td>0.99</td>
<td>0.99</td>
<td>0.92</td>
<td>0.95</td>
<td>0.99</td>
</tr>
<tr>
<td>$\tau_D(\times10^4)$ [s]</td>
<td>1.07</td>
<td>1.65</td>
<td>1.50</td>
<td>1.00</td>
<td>0.80</td>
</tr>
<tr>
<td>$\tau_x/\tau_D$</td>
<td>0.012</td>
<td>0.005</td>
<td>0.005</td>
<td>0.013</td>
<td>0.025</td>
</tr>
</tbody>
</table>
Figure 6. (left) The cross-shore diffusivity $K_x$ versus time $t$ and (right) the non-dimensionalized diffusivity $K_x/\kappa_x$ versus $t/\tau_x$ for the 5 HB06 release days (rows). In each panel, the observed (dashed), LSM modeled (solid), and the analytic (dash-dot) cross-shore diffusivities are shown. The shaded gray region represents the observed $K_x^{(o)}$ sampling error. For the right column, panels (a2)-(e2), the best-fit $\tau_x$ and $\kappa_x = \sigma_u^2 \tau_x$ are used to scale time and the diffusivity respectively, and the analytic expression for the long-time $K_x/\kappa_x = (1 - 2/\pi) \approx 0.36$ is shown (thin dashed line).
Figure 7. Alongshore diffusivity $K_y$ versus time for the five release days. In each panel the observed (dashed), LSM modeled (thick solid) and the LSM with $V = 0$ (thin solid) are indicated by the legend.
Figure 8. The shear-induced observed $K_{S}^{(o)}$ (thin solid curve), LSM modeled $K_{S}^{(m)}$ (thick solid curve), and analytic $K_{S}^{(a)}$ (Eq. 18, dashed curve) alongshore diffusivity versus time for each release day.
Figure 9. Drifter-derived versus ADV-derived (a) $u^2(x)$ and (b) $v^2(x)$. The drifter velocity variances are interpolated to the ADV locations. The thin black line represents the 1:1 relationship and the best-fit line is dashed-black. In (a) the best-fit slope $m = 1.3$, intercept $b = 0.009$, and correlation $r = 0.72$, and in (b) the best-fit slope $m = 1.9$, $b = 0.007$ and correlation $r = 0.67$. 